Closed-Form Upper and Lower Bounds on Coverage Probability of Repulsive Wireless Networks

Jeong Seon Yeom, *Graduate Student Member, IEEE*, Eunmi Chu, *Member, IEEE*, Jong Min Kim, *Member, IEEE*, and Bang Chul Jung, *Senior Member, IEEE*

***Abstract*—In this paper, we propose novel *closed-form* lower and upper bound expressions on coverage probability of repulsive wireless networks modeled by Matèrn hard-core process type II (MHCP-II). The derived mathematical expressions are matched well with simulation results especially when the hard-core distance, δ, of MHCP-II model is larger than the communication distance between transmitter and receiver, R. It is worth noting that the case that $\delta \geq R$ is very important in practical repulsive wireless networks. We also propose a novel closed-form expression on coverage probability of in-homogeneous Poisson point process model in order to approximate the coverage probability of repulsive wireless networks when $\delta < R$. Through extensive computer simulations, we validate that the proposed mathematical analysis matches well with simulation results in various system parameters. To the best of our knowledge, the proposed closed-form expressions are the first mathematically tractable and effective theoretical results in the literature.**

***Index Terms*—Coverage probability, Matérn hard-core process, Poisson point process, repulsive wireless networks, stochastic geometry.**

I. Introduction

Stochastic geometry (SG) framework has received much interest due to its excellence in characterizing large-scale system-level performances of wireless networks [1]. Recently, the SG framework was exploited to mathematically analyze various emerging wireless networks such as intelligent reflection surface (IRS)-assisted networks [2], beyond 5G cellular networks [3], and low-earth-orbit (LEO) satellite communication networks [4]. Poisson point process (PPP) has been widely used to model the spatial distribution of communication nodes in wireless networks, where the nodes are assumed to be distributed independently to each other over communication area. In practical wireless networks, however, there exists a *repulsive* characteristics among communication nodes especially in CSMA ad hoc networks, heterogeneous cellular networks, or vehicular networks, which is modeled by hard-core point process models in general [5]. Matèrn hard-core process (MHCP) is one of the most useful mathematical models to capture such a repulsive characteristics in wireless networks [6], [7]. There exist two popular MHCP models in the literature, called MHCP type I and II, where both MHCP models are derived by applying a *dependent* thinning rule to a parent PPP model. MHCP type I (MHCP-I) eliminates all pairs of points within a certain pairwise (i.e., hard-core) distance δ, while in MHCP type II (MHCP-II) each node has a random associated mark and a node is eliminated only if there exists another node having a smaller mark within the distance. In particular, MHCP-II has been actively investigated since it captures well the repulsive characteristics of wireless networks, and thus we focus on MHCP-II in this paper.

In order to characterize the spatial correlation of MHCP-II model, it is required to obtain a mathematically tractable and closed-form pair correlation function (PCF) among points. However, mathematical analysis of wireless networks modeled by the MHCP is very challenging, and thus many studies only derive approximations on the mean or the moment generating function of interference. For instance, the mean interference in MHCP-II model was mathematically investigated with the lower bounded PCFs [8]. More accurate analysis of the mean interference in MHCP-II model was proposed by approximating the PCF with affine functions, but the accuracy of analytical results becomes deteriorated as the hard-core distance increases [9]. Recently, the interference characteristics of MHCP-II model was mathematically derived in terms of variance, covariance, and correlation even though their final expressions still involve multiple integrals [10]. In [11], a lower bound on coverage probability of MHCP-II model was derived, but the lower bound still involves integrals, which is not a closed-form expression. Due to mathematical intractability, the coverage probability of repulsive wireless networks modeled by MHCP-II was obtained with multiple integrals so far [12].

On the other hand, the *in-homogeneous* PPP approximation was considered to characterize aggregate interference in MHCP-II model [5], [8], where it is assumed that there exist no communication nodes within hard-core region and nodes are distributed according to PPP outside the hard-core region. It is worth noting that there is no closed-form expression for coverage probability even if the approximated in-homogeneous PPP model is adopted instead of MHCP-II model. Thus, many studies adopted the *homogeneous* PPP model with the same node intensity as a simpler PPP approximation to derive closed-form coverage probability, but it does not match well with exact simulation results in general. Furthermore, the interference dynamics of MHCP-II model is significantly different from the PPP approximation models as discussed in [10].

In this paper, we propose *closed-form* lower and upper bounds on the coverage probability of wireless networks based on MHCP-II model, which are mathematically tractable. To the best of our knowledge, it is the first theoretical result in the literature. We show that the bounds proposed in this paper are quite tight especially when the communication range is smaller than the hard-core distance through computer simulations. We propose another closed-form (approximation) mathematical expression on the coverage probability for MHCP-II model by regarding the MHCP-II model as *in-homogeneous PPP* model. Our in-homogeneous PPP approximation expression provides a quite accurate estimate of the coverage probability when the hard-core distance is similar to or smaller than the communication distance.

The rest of this paper is organized as follows. We describe the considered system model in Section II, and we mathematically analyze the coverage probability of MHCP-II model in Section III. In Section IV, we validate our analytical results via extensive computer simulations

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Jeong Seon Yeom and Bang Chul Jung are with the Department of Electronics Engineering, Chungnam National University, Daejeon 34134, South Korea (e-mail: jsyeom@cnu.ac.kr; bcjung@cnu.ac.kr).

Eunmi Chu is with the Internet of Things Examination Division, Korean Intellectual Property Office, Daejeon 35208, South Korea (e-mail: emchu@korea.kr).

Jong Min Kim is with the Department of Mathematics and Information Science, Korea Science Academy of KAIST, Busan 47162, South Korea (e-mail: franzkim@gmail.com).

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TABLE I
MATHEMATICAL NOTATIONS AND DESCRIPTIONS

Notation	Description
Φ_p	Stationary PPP with intensity λ_p
Φ	Stationary MHCP-II with intensity λ
R	Communication distance
δ	Hard-core distance
P_t	Transmit power
h	Small scale fading channel
α	Path-loss exponent
I_o	Received aggregate interference at typical receiver
η	Noise power
θ	SINR threshold

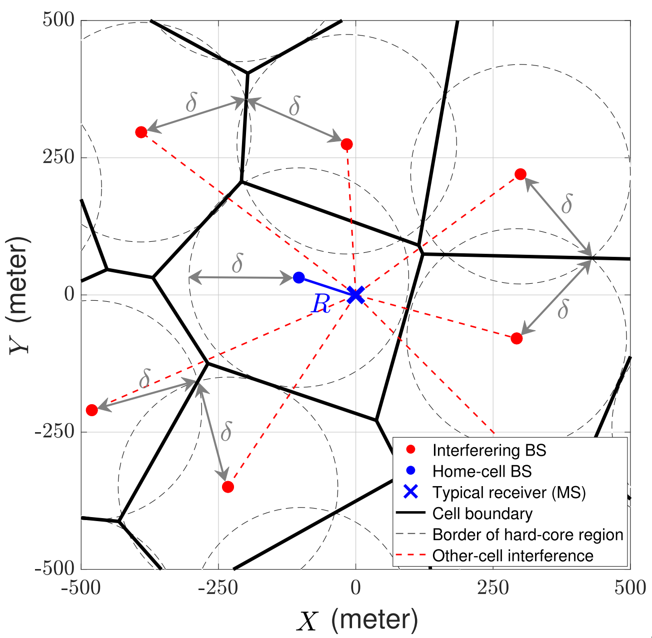

Fig. 1. Downlink cellular network as a representative example of MHCP-II model.

with numerical examples. Finally, conclusions are drawn in Section V. We summarize mathematical notations that will be used in this paper in Table I.

II. SYSTEM MODEL

All transmitting nodes are first distributed according to a stationary PPP, $\Phi_p = \{x_1, x_2, \ldots\}$, with intensity λ_p on $\mathbb{R}^2$, and each transmitter x_i is associated with a random mark uniformly distributed over (0,1). According to MHCP-II model, any transmitter cannot transmit signals if there exists another transmitter within the hard-core distance δ except for the transmitter with the smallest mark. Fig. 1 illustrates the Voronoi diagram of cellular downlink networks as an representative example of MHCP-II model, where base stations (BSs) are distributed according to MHCP-II model with hard-core distance δ. The blue point indicates the home-cell BS and red points indicate other-cell BSs. The distance from the home-cell BS to a typical receiver is denoted by R, which tends to be smaller than the hard-core distance, i.e., $R \leq \delta$ as illustrated in Fig. 1. In particular, R becomes much smaller than δ, $R \ll \delta$, as the typical receiver is located in the cell-center.

The resultant intensity of MHCP-II model becomes $\lambda = \frac{1-e^{-\lambda_p \pi \delta^2}}{\pi \delta^2}$ [13]. For a given distance between arbitrary two points r, we consider the PCF $g(r)$ for MHCP model, which is obtained by normalizing the second-order product density $\rho^{(2)}(r)$ by λ^2 as follows:

$$g(r) = \frac{\rho^{(2)}(r)}{\lambda^2}. \tag{1}$$

The term $\rho^{(2)}(r)$ represents the mean value of the number of all point pairs that are separated by a distance r. For MHCP-II model, $\rho^{(2)}(r) = \lambda_p^2 k(r)$, where $k(r)$ denotes the probability that a point pair with the distance r is retained in Φ_p as follows [8]:

$$k(r) = \frac{2V_\delta(r)\left(1 - e^{-\lambda_p \pi \delta^2}\right) - 2\pi\delta^2\left(1 - e^{-\lambda_p V_\delta(r)}\right)}{\lambda_p^2 \pi \delta^2 V_\delta(r)\left(V_\delta(r) - \pi\delta^2\right)}, \tag{2}$$

where $V_\delta(r)$ denotes an area of union of two circles with radius δ at two points separated by a distance r and is given by

$$V_\delta(r) = 2\pi\delta^2 - 2\delta^2 \arccos\left(\frac{r}{2\delta}\right) + \frac{r}{2}\sqrt{4\delta^2 - r^2}, \tag{3}$$

where $0 \leq r < 2\delta$. For $r \geq 2\delta$, $\rho^{(2)}(r) = \lambda^2$ and $g(r) = 1$ since the union area does not have intersection. For $0 \leq r < \delta$, in contrast, two points become repulsive to each other, $g(r) = 0$.

We assume that the typical receiver is located at the center point o and its corresponding transmitter $x_o \in \Phi$ is assumed to be exist at a distance R from the typical receiver. Then, the desired signal power at the typical receiver is given by $P_t h_o R^{-\alpha}$, where P_t denotes a transmit power from the transmitter, h_o denotes the Rayleigh fading channel gain, i.e., $h_o \sim \exp(1)$, and α denotes the path-loss exponent. The aggregate interference at the typical receiver is given by

$$I_o = \sum_{x_i \in \Phi \setminus \{x_o\}} P_t h |x_i|^{-\alpha}, \tag{4}$$

where h denotes the Rayleigh fading channel gain, i.e., $h \sim \exp(1)$ and $|x_i|$ denotes the distance from the origin to the i-th interferer. The signal-to-interference-plus-noise ratio (SINR) at the typical receiver is given by

$$\mathsf{SINR} = \frac{P_t h_o R^{-\alpha}}{I_o + \eta}, \tag{5}$$

where η denotes the noise power at the typical receiver.

III. COVERAGE PROBABILITY ANALYSIS

In this section, we mathematically analyze the coverage probability of MHCP-II model. The coverage probability is defined as the probability that the SINR is greater than a certain threshold θ. Without loss of generality we assume that the transmit power is unity, i.e., $P_t = 1$. Then, the coverage probability of typical receiver is given by

$$\mathcal{P}_c = \Pr\{\mathsf{SINR} \geq \theta\} = \Pr\left\{\frac{h_o R^{-\alpha}}{I_o + \eta} \geq \theta\right\}. \tag{6}$$

The coverage probability can be rewritten by [1]

$$\begin{aligned} \Pr\left\{\frac{h_o R^{-\alpha}}{I_o + \eta} \geq \theta\right\} &= \Pr\left\{h_o R^{-\alpha} \geq \theta\left(I_o + \eta\right)\right\} \\ &= \Pr\left\{h_o \geq (I_o + \eta)\theta R^\alpha\right\} \\ &= e^{-\eta\theta R^\alpha} \mathbb{E}_o^!\left[-e^{I_o \theta R^\alpha}\right]. \end{aligned} \tag{7}$$

Let $s = \theta R^{\alpha}$. With Campbell's theorem the second term in (7) is given by [1]

$$\begin{aligned}\mathcal{L}_{I_o}(s) &= \mathbb{E}_o^!\left[e^{-sI_o}\right] = \mathbb{E}_{\Phi,h}\left[\exp\left(-s\sum_{x_i\in\Phi\setminus\{o\}} h\|x_i\|^{-\alpha}\right)\right]\\ &= \mathbb{E}_{\Phi}\left[\prod_{x_i\in\Phi\setminus\{o\}}\mathbb{E}_h\left(e^{-sh\|x_i\|^{-\alpha}}\right)\right]\\ &= \exp\left\{-\int_0^{\infty} 2\pi\left(1-\mathbb{E}_h\left[e^{-shr^{-\alpha}}\right]\right) r\lambda g(r)dr\right\}, \quad (8)\end{aligned}$$

where $g(r)$ denotes the PCF of MHCP-II model.

In the following two subsections, closed-form mathematical expressions on coverage probability are derived by using mathematically tractable PCF bounds, which yield high accuracy especially in the relatively large hard-core distance regime. Then, a novel closed-form approximation of coverage probability is derived by regarding the MHCP-II model as an in-homogeneous PPP model for the relatively short hard-core distance regime.

A. Closed-Form Lower Bound

Theorem 1 (Lower Bound): Let $\nu = 12\pi/(8\pi + 3\sqrt{3})$. The lower bound of the coverage probability of MHCP-II model is given by

$$\begin{aligned}\underline{\mathcal{P}}_c = e^{-\eta\theta R^{\alpha}}\exp\Bigg[&-\pi\lambda\frac{\theta R^{\alpha}\delta^{2-\alpha}}{\left(2-\frac{2}{\alpha}\right)}\left\{\frac{1}{1-\frac{2}{\alpha}}+\left(1+\theta R^{\alpha}\delta^{-\alpha}\right)^{-2}\right\}\\ &-2\pi\lambda(\nu-1)\frac{\theta R^{\alpha}\delta^{2-\alpha}}{\left(2-\frac{1}{\alpha}\right)}\left\{\frac{1}{1-\frac{1}{\alpha}}+\left(1+\theta R^{\alpha}\delta^{-\alpha}\right)^{-2}\right\}\\ &+\pi\lambda\frac{\theta R^{\alpha}\delta^{2}}{\delta^{\alpha}+\theta R^{\alpha}}\left(2\nu-1\right)\Bigg]. \quad (9)\end{aligned}$$

Proof: For deriving the lower bound for the coverage probability, we first try to obtain a mathematically tractable upper bound of the PCF $g(r)$, which is denoted by $h(r)$. In order for $h(r)$ to operate as an upper bound of the PCF $g(r)$, it needs to satisfy the following conditions: 1) its slope for r in $[\delta, 2\delta)$ is greater than $\frac{1-\nu}{\delta}$ if $h(\delta) = \nu$ and 2) $\lim_{r\to\infty} h(r) = 1^{+}$ since $g(r) = 1$ for $r \geq 2\delta$. Based on the above two conditions and similar to [9], we propose a novel upper bound of PCF as follows:

$$h(r) = \frac{(\nu-1)\delta}{r} + 1. \quad (10)$$

Applying $h(r)$ instead of $g(r)$ to analyze the coverage probability, a lower bound of (8) $\underline{\mathcal{L}}_{I_o}(s)$ is obtained by

$$\begin{aligned}&\mathcal{L}_{I_o}(s) \geq \underline{\mathcal{L}}_{I_o}(s)\\ &\quad= \exp\left\{-\int_{\delta}^{\infty}\left(1-\mathbb{E}_h\left[e^{-shr^{-\alpha}}\right]\right)2\pi r\lambda\left(\frac{(\nu-1)\delta}{r}+1\right)dr\right\}. \quad (11)\end{aligned}$$

We can reverse the order of integrals in (11) as $\int_a^b \mathbb{E}_y[f(x,y)]dx = \mathbb{E}_y[\int_a^b f(x,y)dx]$ if variables x and y are independent and a and b are constant or infinity. In order to solve an integral form as $\int_{\delta}^{\infty}(1-\exp(-shr^{-\alpha}))\zeta_{1+q}r^{q}dr$ for $\underline{\mathcal{L}}_{I_o}(s)$ where $q \in \{0, 1\}$, we consider the following relationship:

$$\begin{aligned}&\zeta_{1+q}\int_{\delta}^{\infty}\left(1-e^{-shr^{-\alpha}}\right)r^{q}dr\\ &= \frac{\zeta_{1+q}}{1+q}\left[(sh)^{\frac{1+q}{\alpha}}\int_0^{sh\delta^{-\alpha}} e^{-t}t^{-\frac{1+q}{\alpha}}dt - \left(1-e^{-sh\delta^{-\alpha}}\right)\delta^{1+q}\right]\\ &= \frac{\zeta_{1+q}}{1+q}\left[(sh)^{\frac{1+q}{\alpha}}\gamma\left(1-\frac{1+q}{\alpha}, sh\delta^{-\alpha}\right) - \left(1-e^{-sh\delta^{-\alpha}}\right)\delta^{1+q}\right], \quad (12)\end{aligned}$$

where $t \leftarrow r^{-\alpha}sh$ and $\gamma(a,x)$ denotes the lower incomplete gamma function $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$.

Let $p = 1+q$ and then the expectation of (12) for the channel fading h is expressed as follows:

$$\begin{aligned}&\mathbb{E}_h\left[\zeta_{1+q}\int_{\delta}^{\infty}\left(1-e^{-shr^{-\alpha}}\right)r^{q}dr\right]\\ &= \frac{\zeta_p}{p}\mathbb{E}_h\left[(sh)^{\frac{p}{\alpha}}\gamma\left(1-\frac{p}{\alpha}, sh\delta^{-\alpha}\right) - \left(1-e^{-sh\delta^{-\alpha}}\right)\delta^{p}\right]\\ &= \frac{\zeta_p}{p}s^{\frac{p}{\alpha}}\mathbb{E}_h\left[h^{\frac{p}{\alpha}}\gamma\left(1-\frac{p}{\alpha}, sh\delta^{-\alpha}\right)\right] - \frac{\zeta_p}{p}\frac{\delta^{p}s}{\delta^{\alpha}+s}. \quad (13)\end{aligned}$$

Unfortunately, it is hard to directly obtain a closed-form of the expectation term in (13). Hence, we exploit bounds of the lower incomplete gamma function as follows [14]:

$$\frac{x^{a}}{a}e^{-\frac{ax}{a+1}} \leq \gamma(a,x) \leq \frac{x^{a}}{a(a+1)}(1+ae^{-x}). \quad (14)$$

Note that the condition of a lower bound of $\mathcal{L}_{I_o}(s)$ is equivalent to the condition of an upper bound of (13). Then, the first term in (13) has the upper bound as following:

$$\begin{aligned}&\frac{\zeta_p}{p}s^{\frac{p}{\alpha}}\mathbb{E}_h\left[h^{\frac{p}{\alpha}}\gamma\left(1-\frac{p}{\alpha}, sh\delta^{-\alpha}\right)\right]\\ &\leq \frac{\zeta_p}{p}s^{\frac{p}{\alpha}}\int_0^{\infty} h^{\frac{p}{\alpha}}\frac{(sh\delta^{-\alpha})^{1-\frac{p}{\alpha}}}{(1-\frac{p}{\alpha})(2-\frac{p}{\alpha})}\left\{1+\left(1-\frac{p}{\alpha}\right)e^{-sh\delta^{-\alpha}}\right\}e^{-h}dh\\ &= \frac{\zeta_p}{p}\frac{s\delta^{p-\alpha}}{\left(2-\frac{p}{\alpha}\right)}\left\{\frac{1}{1-\frac{p}{\alpha}}+\left(1+s\delta^{-\alpha}\right)^{-2}\right\}. \quad (15)\end{aligned}$$

The values of ζ_p becomes equal to $2\pi\lambda(\nu-1)\delta$ and $2\pi\lambda$ when $p = 1$ and $p = 2$, respectively.

Finally, we obtain the lower bound (9) from the above equations (7), (11), (13), and (15). ■

B. Closed-Form Upper Bound

Theorem 2 (Upper Bound): The upper bound of the coverage probability of MHCP-II model is given by

$$\begin{aligned}\overline{\mathcal{P}}_c = e^{-\eta\theta R^{\alpha}}\exp\Bigg[&-\pi\lambda\frac{\theta R^{\alpha}\delta^{2-\alpha}}{\left(1-\frac{2}{\alpha}\right)}\left\{1+\frac{(\alpha-2)\,\theta R^{\alpha}\delta^{-\alpha}}{2\alpha-2}\right\}^{-2}\\ &+\pi\lambda\frac{\theta R^{\alpha}\delta^{2}}{\delta^{\alpha}+\theta R^{\alpha}}\Bigg]. \quad (16)\end{aligned}$$

Proof: A lower bound of the PCF can be simply provided by assuming $g(r) = 1$ for $r \geq \delta$. Then, the upper bound of (8), $\overline{\mathcal{L}}_I(s)$, is given by

$$\mathcal{L}_{I_o}(s) \le \overline{\mathcal{L}}_{I_o}(s)$$
$$= \exp\left\{-\int_{\delta}^{\infty}\left(1-\mathbb{E}_h\left[e^{-shr^{-\alpha}}\right]\right)2\pi r\lambda dr\right\}$$
$$\stackrel{(a)}{=} \exp\left\{-\pi\lambda s^{\frac{2}{\alpha}}\mathbb{E}_h\left[h^{\frac{2}{\alpha}}\gamma\left(1-\frac{2}{\alpha}, sh\delta^{-\alpha}\right)\right]+\pi\lambda\frac{\delta^2 s}{\delta^\alpha+s}\right\}, \tag{17}$$

where (a) is derived from the case of $p=2$ in (13).

In contrast to the derivation of the lower bound of coverage probability, the lower bound of lower incomplete gamma function in (14) is exploited for the upper bound of coverage probability. Then the expectation term in (17) is as following:

$$s^{\frac{2}{\alpha}}\mathbb{E}_h\left[h^{\frac{2}{\alpha}}\gamma\left(1-\frac{2}{\alpha}, sh\delta^{-\alpha}\right)\right]$$
$$\ge s^{\frac{2}{\alpha}}\int_0^\infty h^{\frac{2}{\alpha}}\frac{(sh\delta^{-\alpha})^{1-\frac{2}{\alpha}}}{\left(1-\frac{2}{\alpha}\right)}e^{-\frac{\left(1-\frac{2}{\alpha}\right)sh\delta^{-\alpha}}{2-\frac{2}{\alpha}}}e^{-h}dh$$
$$= \frac{s\delta^{2-\alpha}}{\left(1-\frac{2}{\alpha}\right)}\int_0^\infty he^{-h\left\{1+\frac{(\alpha-2)s\delta^{-\alpha}}{2\alpha-2}\right\}}dh$$
$$= \frac{s\delta^{2-\alpha}}{\left(1-\frac{2}{\alpha}\right)}\left\{1+\frac{(\alpha-2)\,s\delta^{-\alpha}}{2\alpha-2}\right\}^{-2}. \tag{18}$$

Finally, we obtain the upper bound of coverage probability in (16) from the equations (7), (17), and (18).

C. Closed-Form PPP Approximations

As noted before, many studies have adopted PPP approximations to mathematically characterize the coverage probability of MHCP-II model. The simplest PPP approximation is to use the homogeneous PPP model with the same node density as MHCP-II model, where the closed-form coverage probability is given by [1]

$$\mathcal{P}_c^{\mathsf{PPP}} = e^{-\eta\theta R^\alpha}\exp\left\{-\pi\lambda\Gamma\left(1-\frac{2}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right)\theta^{\frac{2}{\alpha}}R^2\right\}. \tag{19}$$

Another PPP approximation method is to use in-homogeneous PPP model for obtaining the coverage probability of MHCP-II model [5], [8]. Unfortunately, there exists no closed-form expression of the coverage probability in in-homogeneous PPP model as well. Thus, in this paper, we propose a novel closed-form (lower-bound) expression on coverage probability of in-homogeneous PPP model. Then, we can use the following theorem as a closed-form approximation expression for the coverage probability of MHCP-II model since the in-homogeneous PPP approximation works well.

Theorem 3 (In-homogeneous PPP Approximation): The coverage probability of in-homogeneous PPP is lower-bounded as

$$\mathcal{P}_c^{\mathsf{In\text{-}PPP}} \ge \underline{\mathcal{P}}_c^{\mathsf{In\text{-}PPP}} = \exp\left\{-\pi\lambda\Gamma\left(1-\frac{2}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right)\theta^{\frac{2}{\alpha}}R^2\right\}$$
$$\times e^{-\eta\theta R^\alpha}\exp\left(\pi\lambda\frac{\delta^2\theta R^\alpha}{\delta^\alpha+\theta R^\alpha}\right). \tag{20}$$

Proof: With (7), we analyze the coverage probability of the in-homogeneous PPP model as follows

$$\mathcal{P}_c^{\mathsf{In\text{-}PPP}} = e^{-\eta\theta R^\alpha}\times\mathcal{L}_{I_{\mathsf{In\text{-}PPP}}}(s), \tag{21}$$

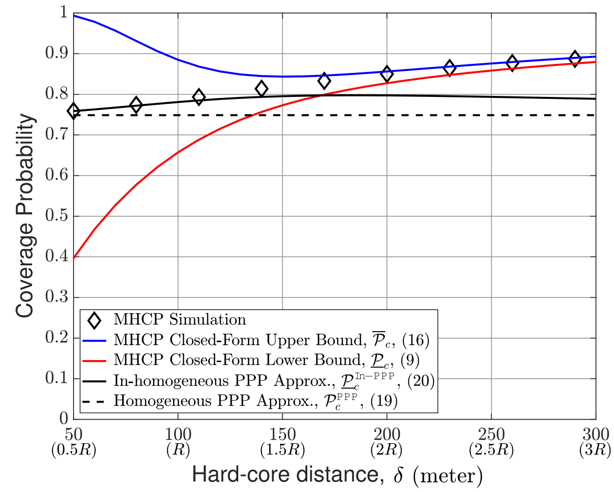

Fig. 2. Coverage probability of MHCP-II model for varying the hard-core distance, δ.

where $\mathcal{L}_{I_{\mathsf{In\text{-}PPP}}}(s)$ denotes the Laplace transform of integrated interference in the in-homogeneous PPP model with intensity λ and hard-core distance δ. By (17), the Laplace transform is rewritten as follow:

$$\mathcal{L}_{I_{\mathsf{In\text{-}PPP}}}(s)$$
$$= \exp\left\{-\int_{\delta}^{\infty}\left(1-\mathbb{E}_h\left[e^{-shr^{-\alpha}}\right]\right)2\pi r\lambda dr\right\}$$
$$= \exp\left\{-\pi\lambda s^{\frac{2}{\alpha}}\mathbb{E}_h\left[h^{\frac{2}{\alpha}}\gamma\left(1-\frac{2}{\alpha}, sh\delta^{-\alpha}\right)\right]+\pi\lambda\frac{\delta^2 s}{\delta^\alpha+s}\right\}$$
$$\stackrel{(b)}{\ge} \exp\left\{-\pi\lambda s^{\frac{2}{\alpha}}\Gamma\left(1-\frac{2}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right)\right\}\exp\left(\pi\lambda\frac{\delta^2 s}{\delta^\alpha+s}\right), \tag{22}$$

where (b) is derived by a lower bound of lower incomplete gamma function, $\gamma(a,x)\le\Gamma(a)$. Substituting $s=\theta R^\alpha$ and (22) into (21), we obtain the lower bound of coverage probability including the (19) in the in-homogeneous PPP model. ■

D. System Optimization Via Analytical Results

Monte-Carlo simulations for large-scaling wireless network with stochastic geometry typically require significantly high computational complexity and time consumption. In order to estimate the performance of repulsive networks, a large amount of computation is required because the distance correlation between points needs be additionally considered and it is not efficient to execute simulations for many situations to set parameters for given performance requirements as well. The proposed closed-form mathematical expressions enable to exactly estimate a large-scale network performance like coverage probability, which can help a system designer to optimize the repulsive wireless networks with a computationally efficient manner.

IV. Numerical Examples

We assume that $R=100\,\text{m}$, $P_t=1\,\text{W}$, and $\alpha=3$ for all simulations in this section. The noise spectral density and the system bandwidth are set to -174 dBm/Hz and 10 MHz, respectively. We consider $70\,\text{km}\times 70\,\text{km}$ square area for whole network. Fig. 2 shows the coverage probability of MHCP-II model for varying the hard-core distance, δ when $\theta=5\,\text{dB}$ and $\lambda=1.7685/\text{km}^2$. We maintain the resultant node intensity by adjusting both λ_p and δ in this figure [13]. We

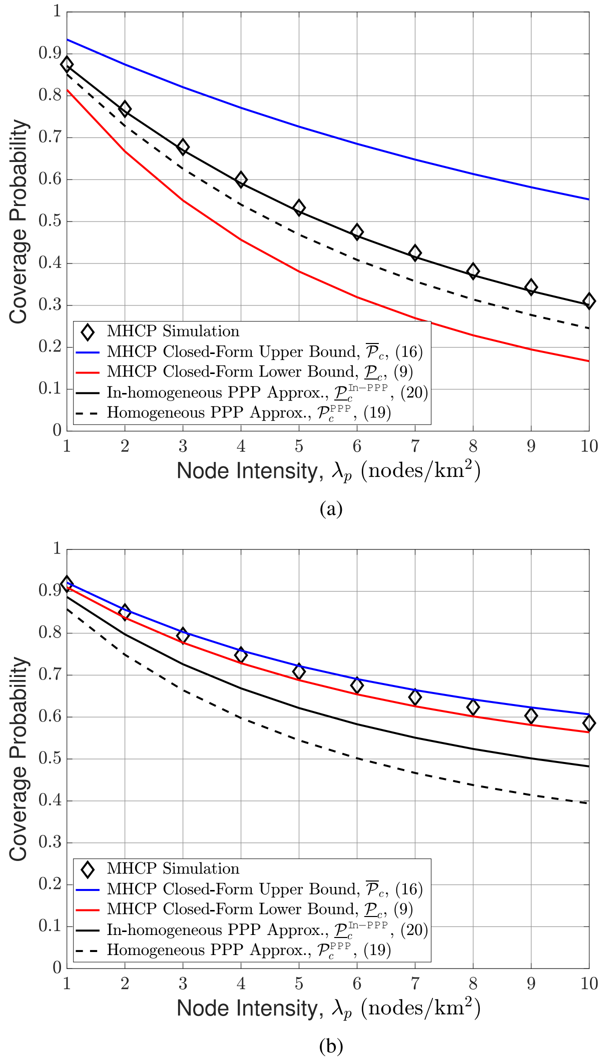

Fig. 3. Coverage probability of MHCP-II model for varying the node intensity, λ_p. (a) $\delta = R$. (b) $\delta = 2R$.

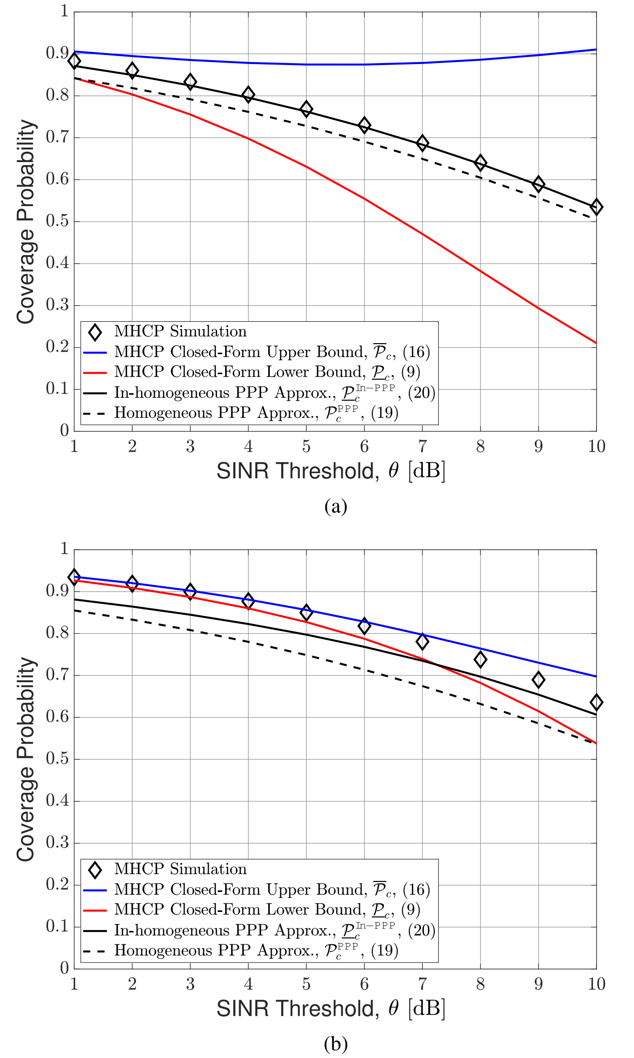

Fig. 4. Coverage probability of MHCP-II model for varying the SINR threshold, θ. (a) $\delta = R$. (b) $\delta = 2R$.

denote the hard-core distance by comparing it with the communication distance for better understanding. When $\delta \geq 2R$, the proposed upper and lower bounds match quite well with the simulation results. On the other hand, the proposed closed-form in-homogeneous PPP approximation matches well when $\delta < 2R$. Thus, we can estimate the coverage probability of repulsive wireless networks quite exactly by using three theorems proposed in this paper according to δ.

Fig. 3 shows the coverage probability of MHCP-II model for varying the node intensity, λ_p, when $\delta = R$, $2R$ and $\theta = 5\,$dB. As expected, the coverage probability decreases as the node density increases. Furthermore, the coverage probability with larger hard-core distance is better than the coverage probability with smaller hard-core distance for a given node density. As shown in the figure, the proposed lower and upper bounds are very tight to each other and match well with the simulation result when $\delta = 2R$ regardless of node intensity. When $\delta = R$, the proposed upper and lower bounds become loose as the node intensity increases while the proposed closed-form in-homogeneous PPP approximation well estimates the coverage probability of MHCP-II model regardless of node intensity.

Fig. 4 shows the coverage probability of MHCP-II model according to the SINR threshold when $\lambda_p = 2/\text{km}^2$ and $\delta = R$, $2R$. The coverage probability decreases as the SINR increases and a larger hard-core distance results in better coverage probability for a given SINR threshold. The proposed lower and upper bounds are very tight to each other and match well with the simulation result when $\delta = 2R$ regardless of SINR threshold. When $\delta = R$, the proposed closed-form in-homogeneous PPP approximation matches well with the simulation result regardless of SINR threshold.

We can optimize the hard-core distance for a given coverage probability in various system parameters by using the proposed mathematical results. Table II compares the minimum required hard-core distance derived from our analytical results with the exact minimum required hard-core distance obtained from the time-consuming Monte-Carlo simulations. The target coverage probabilities are set to 0.85, 0.90, and 0.95, while the node intensity varies are assumed to be 1, 1.2, and 1.5. After observing extensive numerical examples including figures in this paper, for determining the minimum required hard-core distance, we propose to use the median value among the proposed lower bound, upper bound, and in-homogeneous PPP approximation in equations (9), (16), and (20), respectively. As shown in Table II, the proposed mathematical analyses estimate a quite exact minimum required hard-core distance to guarantee target coverage probabilities compared with the

TABLE II
MINIMUM REQUIRED HARD-CORE DISTANCE FOR GIVEN COVERAGE PROBABILITIES

λ (/km^2)	Target Coverage Probability					
	0.85		0.9		0.95	
	Simul.	Analysis	Simul.	Analysis	Simul.	Analysis
1	19 m	19 m	164 m	203 m	382 m	382 m
1.2	101 m	116 m	200 m	210 m	461 m	461 m
1.5	156 m	198 m	271 m	273 m	579 m	579 m

simulation results. It is worth noting that a system designer can estimate the exact network performance and optimize system parameters without time-consuming and high-complexity network simulations by using the proposed mathematical analyses.

V. CONCLUSION

In this paper, we proposed three novel closed-form mathematical expressions to analyze the coverage probability of Matérn hard-core process type II model-based repulsive wireless networks: lower-bound, upper-bound, and in-homogeneous Poisson point process (PPP)-based approximation. The lower and upper bounds estimate the coverage probability of repulsive wireless networks very accurately when the hard-core distance is larger than the communication distance, while the in-homogeneous PPP approximation expression provides a quite accurate estimate of the coverage probability when the hard-core distance is similar to or smaller than the communication distance. Therefore, we can exactly predict the coverage probability of repulsive wireless networks with the proposed expressions according to the hard-core distance.

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